Algorithms: Survey of Common Running Times

Why It Matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
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$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Constant time

Constant time. Running time is $O(1)$.

bounded by a constant, which does not depend on input size n

Examples.

- Conditional branch.
- Arithmetic/logic operation.
- Declare/initialize a variable.
- Follow a link in a linked list.
- Access element i in an array.
- Compare/exchange two elements in an array.
- ...

Linear Time: O(n)

Linear time. Running time is proportional to input size.

Computing the maximum. Compute maximum of n numbers $a_1, \ldots, a_n$.

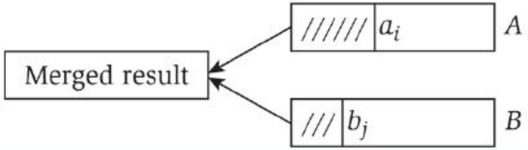
```
max ← a1
for i = 2 to n {
   if (ai > max)
      max ← ai
}
```

Linear time

Linear time. Running time is $O(n)$.

Merge two sorted lists. Combine two sorted linked lists $A = a_1, a_2, \ldots, a_n$ and $B = b_1, b_2, \ldots, b_n$ into a sorted whole.

$O(n)$ algorithm. Merge in mergesort.



$i \leftarrow 1$; $j \leftarrow 1$.

WHILE (both lists are nonempty)

  IF ($a_i \leq b_j$) append a_i to output list and increment i.

  ELSE append b_j to output list and increment j.

Append remaining elements from nonempty list to output list.

Logarithmic time

Logarithmic time. Running time is $O(\log n)$.

Search in a sorted array. Given a sorted array A of n distinct integers and an integer x, find index of x in array.

$O(\log n)$ algorithm. Binary search.

remaining elements

- Invariant: If x is in the array, then x is in $A[lo .. hi]$.
- After k iterations of WHILE loop, $(hi - lo + 1) \le n / 2^k \Rightarrow k \le 1 + \log_2 n$.

```
lo ← 1; hi ← n.
WHILE (lo ≤ hi)
   mid ← ⌊(lo + hi) / 2⌋.
   IF      (x < A[mid])  hi ← mid − 1.
   ELSE IF (x > A[mid])  lo ← mid + 1.
   ELSE RETURN mid.
RETURN −1.
```

Linearithmic time

Linearithmic time. Running time is $O(n \log n)$.

Sorting. Given an array of n elements, rearrange them in ascending order.

$O(n \log n)$ algorithm. Mergesort.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

O(n log n) Time

O(n log n) time. Arises in divide-and-conquer algorithms.

also referred to as linearithmic time

Sorting. Mergesort and heapsort are sorting algorithms that perform O(n log n) comparisons.

Largest empty interval. Given n time-stamps $x_1, \ldots, x_n$ on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

O(n log n) solution. Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.

Quadratic time

Quadratic time. Running time is $O(n^2)$.

Closest pair of points. Given a list of n points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest to each other.

$O(n^2)$ algorithm. Enumerate all pairs of points (with $i < j$).

```
min ← ∞.
FOR i = 1 TO n
   FOR j = i + 1 TO n
      d ← (x_i − x_j)^2 + (y_i − y_j)^2.
      IF (d < min)
         min ← d.
```

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion. [see §5.4]

Cubic Time: $O(n^3)$

Cubic time. Enumerate all triples of elements.

Set disjointness. Given n sets $S_1, \ldots, S_n$ each of which is a subset of 1, 2, …, n, is there some pair of these which are disjoint?

$O(n^3)$ solution. For each pairs of sets, determine if they are disjoint.

```
foreach set Si {
   foreach other set Sj {
      foreach element p of Si {
         determine whether p also belongs to Sj
      }
      if (no element of Si belongs to Sj)
         report that Si and Sj are disjoint
   }
}
```

Cubic time

Cubic time. Running time is $O(n^3)$.

3-SUM. Given an array of n distinct integers, find three that sum to 0.

$O(n^3)$ algorithm. Enumerate all triples (with $i < j < k$).

```
FOR i = 1 TO n
  FOR j = i + 1 TO n
    FOR k = j + 1 TO n
    IF (a_i + a_j + a_k = 0)
      RETURN (a_i, a_j, a_k).
```

Remark. $\Omega(n^3)$ seems inevitable, but $O(n^2)$ is not hard. [see next slide]

Polynomial time

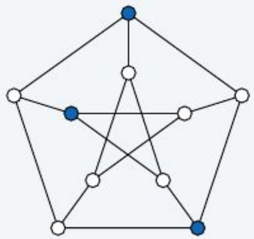
Polynomial time. Running time is $O(n^k)$ for some constant $k > 0$.

Independent set of size k. Given a graph, find k nodes such that no two are joined by an edge.

k is a constant

$O(n^k)$ algorithm. Enumerate all subsets of k nodes.

```
FOREACH subset S of k nodes:
    Check whether S is an independent set.
    IF (S is an independent set)
        RETURN S.
```



independent set of size 3

- Check whether S is an independent set of size k takes $O(k^2)$ time.
- Number of k-element subsets = $\binom{n}{k} = \frac{n(n-1)(n-2) \times \cdots \times (n-k+1)}{k(k-1)(k-2) \times \cdots \times 1} \leq \frac{n^k}{k!}$
- $O(k^2 \, n^k / k!) = O(n^k)$.

poly-time for $k = 17$, but not practical

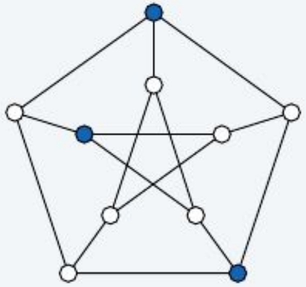
Exponential time

Exponential time. Running time is $O(2^{n^k})$ for some constant $k > 0$.

Independent set. Given a graph, find independent set of max cardinality.

$O(n^2 2^n)$ algorithm. Enumerate all subsets of n elements.

$S^* \leftarrow \varnothing$.
FOREACH subset S of n nodes:
  Check whether S is an independent set.
  IF (S is an independent set and $|S| > |S^*|$)
    $S^* \leftarrow S$.
RETURN S^*.



independent set of max cardinality

Exponential time

Exponential time. Running time is $O(2^{n^k})$ for some constant $k > 0$.

Euclidean TSP. Given n points in the plane, find a tour of minimum length.

$O(n \times n!)$ algorithm. Enumerate all permutations of length n.

$\pi^* \leftarrow \varnothing$.

FOREACH permutation π of n points:

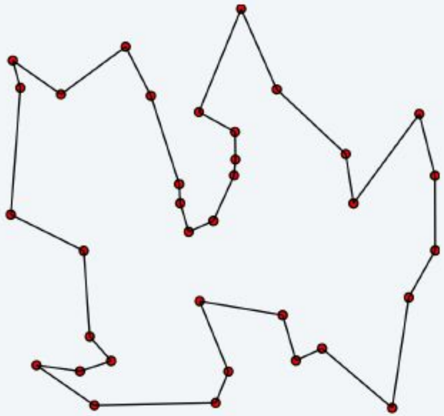
  Compute length of tour corresponding to π.

  IF (length(π) < length(π^*))

    $\pi^* \leftarrow \pi$.

RETURN π^*.

for simplicity, we'll assume Euclidean distances are rounded to nearest integer (to avoid issues with infinite precision)



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Suggested Reading

- Algorithm Design by Jon Kleinberg, Eva Tardos
  - Chapter 2
    - Section: 2.4